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CONSTANT CURVATURE CONDITIONS FOR GENERALIZED KROPINA SPACES

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Abstract: The classification of Finsler spaces of constant curvature is an interesting and important topic of research in differential geometry. In this paper we obtain necessary and sufficient conditions for generalized Kropina space to be of constant flag curvature.

Keywords and Phrases: Riemannian spaces, Killing vector fields, Finsler metrics, Kropina metrics, Generalized Kropina metrics.

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1. Introduction

Finsler metrics are generalization of Riemannian metrics in the sense that they depend on both the position and direction while its counterpart depend only on position. Generalized Kropina metric belongs to the large class of (α, β)-metrics. (α, β)-metrics were firstly introduced by Matsumoto [8]. They are constructed by Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and the differential 1-form $\beta = b_i(x)y^i$. Some remarkable (α, β)-metrices are: Randers metric: F = $\alpha + \beta$; Kropina metric: F = $\frac{\alpha^2}{\beta}$; generalized Kropina metric: F = $\frac{\alpha^{m+1}}{\beta^m}$ (m $\neq$ -1, 0, 1); Matsumoto metric : F = $\frac{\alpha^2}{\alpha-\beta}$ and square metric: F = $\frac{(\alpha+\beta)^2}{\alpha}$. Contrary to other (α, β)-metrics, Kropina

metric and generelised Kropina metric are not regular but they have wide applications in many branches of science.
Classification of (α, β)-metrics is very important problem in Finsler geometry. Several geometers have worked on this topic with different perspective ([2], [7], [9], [13]). The flag curvature is the most significant Riemannian quantity in Finsler geometry because it correlate sectional curvature in Riemannian geometry. The aim of this paper is to find if and only if condition for generalized Kropina space to be of constant flag curvature. Furthermore, Finsler metrics of constant flag curvature are the natural extension of Riemannian metrics of constant sectional curvature.

2. The Description of Generalized Kropina Metric

Let (M, α) be an n($\geq$ 2)-dimensional smooth manifold endowed with Riemannian metric α. A generalized Kropina space $\left(M, \dfrac{\alpha^{m+1}}{\beta^m}\right)$ is a Finsler space whose fundamental function is given by F = $\frac{\alpha^{m+1}}{\beta^m}$ (m $\neq$ -1, 0,1), where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form. For our purpose we assume that the matrix (a_{ij}) is positive definite.

It is to be noted that Randers spaces (M, F = $\alpha + \beta$) on TM are Lagragian duals of Kropina spaces $(M = \bar{F} = \frac{\bar{\alpha}^2}{\beta})$ on T^*M and vice-versa in the case $b^2 = 1$, where b is the Riemannian length of β.

Furthermore, for regular Lagrangians, the necessary and sufficient condition for a Finsler space to be of constant flag curvature K is that its dual space is also of constant flag curvature $\bar{K}$ ([4], [5]). Importance of generalized Kropina metrics can be also seen in dual related problems, L-dual for generalized Kropina spaces and some other Finsler spaces have been obtained in ([10], [11], [12]).

Define a new Reimannian metric h = $\sqrt{h_{ij}(x)y^i y^j}$ and a vector field $W = W^i(\frac{\partial}{\partial x^i})$ on M by [3]

$$h_{ij} = e^{k(x)}a_{ij}, W_i = \frac{1}{2}e^{k(x)}b_i, e^{k(x)}b^2 = 4. \tag{2.1}$$

where $W_i = h_{ij}W^j$.
Then the generalized Kropina metric can be written as

$$F = \frac{\alpha^{m+1}}{\beta^m} = \pi \frac{h_{00}^{\frac{m+1}{2}}}{W_0^m}, \tag{2.2}$$

where $\pi = \frac{e^{\frac{(m-1)k(x)}{2}}}{2^m}$. Riemannian metrics h and α are connected with each other via equation (2.2).

3. The Coefficients of the Geodesic Spray

Let us recall [15], the following theorem for later use:

Theorem 3.1. *Let (M, g) and (M, $g^* = e^{\rho} g$), where $g = \sqrt{g_{ij}(x)y^i y^j}$ and $g^* = \sqrt{g^*_{ij}(x)y^i y^j}$ respectively, be two n-dimensional Riemannian spaces which are conformal to each other. Furthermore, let ${\gamma_j}^i{}_k$ and ${\gamma_j}^{*i}{}_k$ be the coefficients of Levi-Civita connection of (M, g) and (M, g^*), respectively. Then, we have*

$$g^*{}_{ij} = e^{2\rho} g_{ij}, g^{*ij} = e^{-2\rho} g^{ij} \tag{3.1}$$

and

$${\gamma_j}^{*i}{}_k = {\gamma_j}^i{}_k + \rho_j \delta^i_k + \rho_k \delta^i_j - \rho^i g_{jk}, \tag{3.2}$$

where $\rho_i = \frac{\partial \rho}{\partial x^i}$ $\rho^i = g^{ij}\rho_j$.

From (2.1), we have $h_{ij} = e^k a_{ij}$. Applying Theorem 3.1, we get

$${}^h{\gamma_j}^i{}_k ={}^\alpha {\gamma_j}^i{}_k + \frac{1}{2} k_j \delta^i_k + \frac{1}{2} k_k \delta^i_j - \frac{1}{2} k^i a_{jk}, \tag{3.3}$$

where ${}^h{\gamma_j}^i{}_k$ and ${}^\alpha{\gamma_j}^i{}_k$ are the coefficients of levi-Civita connection of (M, h) and (M, α) respectively, $k_i = \partial k/\partial x^i$ and $k^i = a^{ij} k_j$. Transvecting (3.1) by $y^j y^k$, we get

$${}^h\gamma^i_{00} ={}^\alpha \gamma^i_{00} + k_0 y^i - \frac{1}{2} h_{00} \bar{k}^i, \tag{3.4}$$

where $\bar{k}^i = h^{ij} k_j$ and the index 0 means the transvection by y^i.

We denote the covariant derivative in the Riemannian space (M, α) by (;i) and introduce the following notations: $s_{ij} := \frac{b_{i;j} - b_{j;i}}{2}$, $r_{ij} = \frac{b_{i;j} + b_{j;i}}{2}$, $s_j := b^i s_{ij}$.

In [1], authors have shown that the coefficients G^i of geodesic spray in a Finsler space (M, F = $\alpha\phi(s)$), where s = β/α and ϕ is a differential function of s alone, are given by

$$2G^i ={}^\alpha \gamma^i_{00} + 2\omega\alpha s^i_0 + 2\Theta(r_{00} - 2\alpha\omega s_0)\Big(\frac{y^i}{\alpha} + \frac{\omega'}{\omega - s\omega'} b^i\Big), \tag{3.5}$$

where $\omega := \frac{\phi'}{\phi - s\phi'}$, and $\Theta := \frac{\omega - s\omega'}{2\{1 + s\omega + (b^2 - s^2)\omega'\}}$.

For a Generalized Kropina space $\big(M, \frac{\alpha^{m+1}}{\beta^m}\big)$, a new Riemannian metric $h = \sqrt{h_{ij} y^i y^j}$ and a vector field $W = W^i(\partial/\partial x^i)$ are defined by (2.1).

So, using (2.1), we can say that the vector field W satisfies the condition $\|W\| = 1$. For the generalized Kropina metric, we have

$$\begin{aligned}
\phi(s) :=& \frac{1}{s^m}, \\
\omega :=& -\frac{m}{1+m}\frac{1}{s} = -\frac{m}{1+m}\frac{\alpha}{\beta}, \\
\omega' :=& \frac{m}{1+m}\frac{1}{s^2}, \\
\Theta :=& -\frac{ms}{s^2 - ms^2 + mb^2}, \\
s^2 :=& \frac{4{W_0}^2}{e^{k(x)}h_{00}}.
\end{aligned}$$

Furher, we have

$$\Theta := -\frac{mW_0}{2\{(1-m){W_0}^2 + mh_{00}\}}. \tag{3.6}$$

Therefore, we get

$$2G^i =^h \gamma^i_{00} + 2\Phi^i, \tag{3.7}$$

where

$$2\Phi^i : = -k_0 y^i + \frac{1}{2}h_{00}\bar{k}^i + 2\omega\alpha {s^i}_0 + 2\Theta(r_{00} - 2\alpha\omega s_0)(\frac{y^i}{\alpha} + \frac{\omega'}{\omega - s\omega'}b^i). \tag{3.8}$$

Remark. We can introduce a Finsler connection $\Gamma^* = ({}^h\gamma^i_{j\,k}(x), N^i_j : = {}^h\gamma^i_{j\,k}(x)\; y^k,$ $C^i_{j\,k})$ associated with the linear connection ${}^h\gamma^i_{j\,k}(x)$ of the Riemannian space (M, h). The h-covariant derivative are defined as follows [6]:
For a vector field $W^i(x)$ on M,
(1) $W^i(x)_{\|j} = \frac{\partial W^i}{\partial x^j} - \frac{\partial W^i}{\partial y^s}N^s_j + {}^h\gamma^i_{j\,s}W^s = \frac{\partial W^i}{\partial x^j} + {}^h\gamma^i_{j\,s}W^s.$
For a reference vector y^i,
(2) $y^i_{\|j} = \frac{\partial y^i}{\partial x^j} - \frac{\partial y^i}{\partial y^s}N^s_j + {}^h\gamma^i_{j\,s}y^s = -N^j_i + N^j_i = 0.$
We put

$$R_{ij} := \frac{W_{i\|j} + W_{j\|i}}{2}, \quad S_{ij} := \frac{W_{i\|j} - W_{j\|i}}{2}, \quad {R^i}_j := h^{ir}R_{rj}, \quad {S^i}_j = h^{ir}S_{rj},$$
$$s_j := b_j {s^j}_i R_i := W^r R_{ri}, \quad S_i := W^r S_{ri}, \quad R^i := h^{ir}R_r, \quad S^i := h^{ir}S_r.$$

It follows $r_{ij} = 2e^{-k}\big(R_{ij} - \frac{1}{2}W_r\bar{k}^r h_{ij}\big), \quad s_{ij} = 2e^{-k}\big(S_{ij} + \frac{k_iW_j - k_jW_i}{2}\big).$

Furthermore, we get

$$
\begin{aligned}
s^i{}_j &= 2S^i{}_j + \bar{k}^i W_j - k_j W^i, \\
s^i{}_0 &= 2S^i{}_0 + \bar{k}^i W_0 - k_0 W^i, \\
s_i &= 2e^{-k}(2S_i + W_r \bar{k}^r W_i - k_i), \\
s_0 &= 2e^{-k}(2S_0 + W_r \bar{k}^r W_0 - k_0), \\
r_{00} &= 2e^{-k}\big(R_{00} - \frac{1}{2} W_r \bar{k}^r h_{00}\big), \\
b^i &= a^{ir} b_r = e^k h^{ir} \frac{2W_r}{e^k} = 2W^i.
\end{aligned}
$$

Using all these, we get

$$
\begin{aligned}
2\Phi^i = {} & \frac{(m-1)\{m h_{00} + (m+1){W_0}^2\}}{(m+1)\{(1-m){W_0}^2 + m h_{00}\}} k_0 y^i - \frac{(m-1)}{2(m+1)} h_{00} \bar{k}^i - \\
& \frac{m(m-1) h_{00} W_0}{(m+1)\{(1-m){W_0}^2 + m h_{00}\}} k_0 W^i - \frac{m(m-1) W_r \bar{k}^r h_{00}}{2(m+1)\{(1-m){W_0}^2 + m h_{00}\}} \\
& (2W_0 y^i - h_{00} W^i) - \frac{2m}{m+1} \frac{h_{00}}{W_0} S^i{}_0 - \frac{m}{(1-m){W_0}^2 + m h_{00}} \\
& (R_{00} + \frac{2m}{m+1} \frac{h_{00}}{W_0} S_0)(2W_0 y^i - h_{00} W^i). \qquad (3.9)
\end{aligned}
$$

Remark. Putting m = 1 in above equation, we have

$$
2\Phi^i = \frac{h_{00}}{W_0}(S_0 W^i - S^i{}_0) + (R_{00} W^i - 2S_0 y^i) - \frac{2W_0}{h_{00}} R_{00} y^i.
$$

The above equation coincides with (2.6) in [14].

Using (3.9), we can obtain

$$
2\Phi^i h_{00}^{\frac{m+1}{2}} W_0^m = A^i_{(1)} h_{00}^{\frac{m+3}{2}} + A^i_{(2)} h_{00}^{\frac{m+1}{2}} W_0^m + A^i_{(3)} W_0^{m+2} h_{00}^{\frac{m-1}{2}},
$$

or

$$
2\Phi^i h_{00} W_0^m = A^i_{(1)} h_{00}^2 + A^i_{(2)} h_{00} W_0^m + A^i_{(3)} W_0^{m+2}, \qquad (3.10)
$$

where

$$A^i_{(1)} =\sigma_0\{m(m-1)W_0\bar{k}^r - 2m^2S_0W^i\}{W_0}^{m-1},$$

$$A^i_{(2)} =\sigma_0 m(m-1) - \frac{(m-1)}{2(m+1)}\bar{k}^i - \frac{m(m-1)\sigma_0 k_0 W^i}{W_0} - \sigma_0 m(m-1)W_0 W_r \bar{k}^r y^i$$
$$- \frac{2m}{(m+1)W_0}{S^i}_0 + m(m+1)R_{00}W^i - 4\sigma_0 m^2 S_0 y^i,$$

$$A^i_{(3)} =\sigma_0\{(m^2-1) - \frac{2m(m+1)}{W_0}R_{00}y^i\},$$

and $\sigma_0 =\dfrac{1}{(m+1)\{(1-m)\frac{W_0^2}{h_{00}} + m\}}$.

4. The Necessary and Sufficient Conditions for Constant Curvature of Generalized Kropina Spaces

In this section, we consider a Generalized Kropina space (M, α^{m+1}/β^m) of constant curvature K, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differentiable 1-form. Furthermore, we suppose that the matrix (a_{ij}) is always positive definite and that the dimension $n \geq 2$. Hence, it follows that α^{m+1} is not divisible by β^m. This is an important relation and it is equivalent to that $h_{00}^{\frac{m+1}{2}}$ is not divisible by W_0^m. Using these, we shall obtain the necessary and sufficient conditions for a Kropina space to be of constant curvature.

4.1. The Curvature Tensor of a Generalized Kropina Space

Let ${R_j}^i{}_{kl}$ be the h-curvature tensors of Cartan connection in Finsler space. The Berwald spray curvature tensor is

$$^{(b)}{R_j}^i{}_{kl} = A_{(kl)}\Big(\frac{\partial {G_j}^i{}_k}{\partial x^l} + {G_j}^r{}_k {G_r}^i{}_l\Big), \tag{4.1}$$

where the symbol $A_{(kl)}$ denotes the interchange of indices k and l and subtraction. It is well known that the equality ${R_0}^i{}_{kl} =^{(b)} {R_0}^i{}_{kl}$ holds good [15].

From $2G^i =^h{\gamma_0}^i{}_0 + 2\Phi^i$, it follows $G^i_j =^h {\gamma_0}^i{}_j + {\Phi_j}^i{}_k$, where $\Phi^i_j := \frac{\partial \Phi^i}{\partial y^j}$ and ${\Phi^i}_{jk} := \frac{\partial \Phi^i_j}{\partial y^k}$. Substituting the above equalities in (4.1), we get

$^{(b)}{R_j}^i{}_{kl} =^h {R_j}^i{}_{kl} + A_{(kl)}\{{\Phi_j}^i{}_{k\|l} + {\Phi_j}^r{}_k {\Phi_r}^i{}_l\}$.

The following results are well known [14]:

Proposition 4.1. *The necessary and sufficient condition for a Finsler space (M, F) to be of scalar curvature K is that the equality*

$${R_0}^i{}_{0l} = KF^2(\delta^i_l - l^il_l), \tag{4.2}$$

where $l^i = y^i/F$ *and* $l_l = \partial F/\partial y^l$*, holds.*

If the equality (4.2) holds and K is constant, then the Finsler space is said to be of constant curvature K.

For a generalized Kropina space of constant curvature K, since $F = \varepsilon h_{00}^{\frac{m+1}{2}}/(2W_0)^m$, where $\varepsilon = (e^{k(x)})^{\frac{m-1}{2}}$, we have

$l^i l_l = \frac{\frac{m+1}{2}W_0 h_{0l} - m h_{00} W_l}{W_0 h_{00}} y^i.$

So,

$$\delta^i_l - l^i l_l = \delta^i_l - \frac{\frac{m+1}{2}W_0 h_{0l} - m h_{00} W_l}{W_0 h_{00}} y^i.$$

Using the curvature obtained above, we have $R_0{}^i{}_{0l} = {}^h R_0{}^i{}_{0l} + 2\Phi^i_{\|l} - \Phi^i_{l\|0} + 2\Phi^r \Phi_r{}^i{}_l - \Phi^r_l \Phi^i_r$.

Substituting the above equalities in (4.2), we get

$$K \frac{\varepsilon^2 h_{00}^{m+1}}{(2W_0)^{2m}} h^i{}_l = {}^h R_0{}^i{}_{0l} + 2\Phi^i_{\|l} - \Phi^i_{l\|0} + 2\Phi^r \Phi_r{}^i{}_l - \Phi^r_l \Phi^i_r. \tag{4.3}$$

4.2. Rewriting the equation (4.3) using h_{00} and W_0

(1). The calculation for $\Phi^i{}_{\|l}$.

First, applying the h-covariant derivative $_{\|l}$ to (3.12), it follows:

$$\begin{aligned} 2h_{00}W_0^m \Phi^i_{\|l} + 2h_{00} m W_0{}^{m-1} W_{0\|l} \Phi^i = {} & h_{00}^2 A^i_{(1)\|l} + h_{00} m W_0{}^{m-1} W_{0\|l} A^i_{(2)} + \\ & (h_{00}) W_0^m A^i_{(2)\|l} + W_0^{m+2} A^i_{(3)\|l} + \\ & (m+2) W_0^{m+1} W_{0\|l} A^i_{(3)}, \end{aligned}$$

again using (3.12), we have

$$\begin{aligned} 2h_{00}W_0^{m+1}\Phi^i_{\|l} = {} & h_{00}^2 W_0 A^i_{(1)\|l} - m h_{00}^2 W_{0\|l} A^i_{(1)} + h_{00} W_0^{m+1} A^i_{(2)\|l} \\ & + W_0^{m+3} A^i_{(3)\|l} + 2W_0^{m+2} W_{0\|l} A^i_{(3)}. \end{aligned}$$

By appropriate substitutions, we get

$$\begin{aligned} 2h_{00}W_0^{m+1}\Phi^i_{\|l} = {} & h_{00}^2 W_0 B^i_{(1)\|l} + m h_{00}^2 B^i_{(21)l} + h_{00} W_0^{m+1} B^i_{(22)l} \\ & + W_0^{m+3} B^i_{(3)l} + 2W_0^{m+2} B^i_{(4)l}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} B^i_{(1)\|l} =& A^i_{(1)\|l}, \\ B^i_{(21)l} =& -W_{0\|l}A^i_{(1)}, \\ B^i_{(22)l} =& A^i_{(2)\|l}, \\ B^i_{(3)l} =& A^i_{(3)\|l}, \\ B^i_{(4)l} =& W_{0\|l}A^i_{(3)}. \end{aligned}$$

(2). The calculation for Φ^i_l.

Secondly, differentiating equation (3.10) by y^l, we get

$$\begin{aligned} 2\Phi^i_l h_{00}^{\frac{m+3}{2}} W_0^{m+1} = & h_{00}^{m+2}W_0C^i_{(0)l} + h_{00}^{m+2}C^i_{(11)l} + h_{00}^{\frac{m+3}{2}}W_0^{m+1}Ci_{(12)l} + \\ & (h_{00})^{m+1}W_0C^i_{(21)l} + h_{00}W_0^{2m+1}C^i_{(22)l} + W_0^{2m}h_{00}C^i_{(3)l} + \\ & W_0^{2m+1}C^i_{(4)l}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} C^i_{(0)l} &= A^i_{(1)l}, \\ C^i_{(11)l} &= -mW_lA^i_{(1)}, \\ C^i_{(12)l} &= A^i_{(2)l}, \\ C^i_{(21)l} &= (m+1)h_{0l}A^i_{(1)}, \\ C^i_{(22)l} &= A^i_{(3)l}, \\ C^i_{(3)l} &= mW_lA^i_{(3)}, \\ C^i_{(4)l} &= -(m+1)h_{0l}A^i_{(3)}. \end{aligned}$$

(3). The Calculation for $\Phi^i_{l\|0}$.
Applying the h-covariant derivative $_{\|0}$ to (4.5), we get

$$\begin{aligned} 2h_{00}^{\frac{m+3}{2}}W_0^{m+2}\Phi^i_{l\|0} =& h_{00}^{m+2}W_0^2D^i_{(1)l} + h_{00}^{m+2}W_0D^i_{(21)l} + \\ & h_{00}^{m+2}D^i_{(31)l} + h_{00}^{\frac{m+3}{2}}W_0^{m+2}D^i_{(22)l} \\ & + h_{00}^{m+1}W_0^2D^i_{(32)l} + h_{00}^{m+1}W_0D^i_{(41)l} \\ & + h_{00}W_0^{2m+2}D^i_{(33)l} + h_{00}W_0^{2m+1}D^i_{(42)l} + \\ & W_0^{2m+2}D^i_{(5)l} + W_0^{2m+1}D^i_{(6)l}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
D^i_{(1)l} &= C^i_{(0)l\|0},\\
D^i_{(21)l} &= C^i_{(11)l\|0} - W_{0\|0}C^i_{(0)l},\\
D^i_{(31)l} &= -2W_{0\|0}C^i_{(11)l},\\
D^i_{(22)l} &= C^i_{(12)l\|0}, D^i_{(32)l} = C^i_{(21)l\|0},\\
D^i_{(41)l} &= -W_{0\|0}C^i_{(21)}, D^i_{(33)l} = C^i_{(22)l\|0},\\
D^i_{(42)l} &= W_{0\|0}C^i_{(22)l} + C^i_{(3)l\|0}, D^i_{(5)l} = C^i_{(4)l\|0}, D^i_{(6)l} = W_{0\|0}C^i_{(4)l}.
\end{aligned}$$

(4). The Calculation for $\Phi^r_l\Phi^i_r$

$$\begin{aligned}
&4\Phi^r_l\Phi^i_r(h_{00})^{m+3}W_0^{2m+2}\\
&= (h_{00})^{2m+4}W_0^2E^i_{(01)l} + (h_{00})^{2m+4}W_0E^i_{(11)l} + (h_{00})^{2m+4}E^i_{(21)l}+\\
&+ (h_{00})^{\frac{3m+7}{2}}W_0^{m+2}E^i_{(12)l} + (h_{00})^{\frac{3m+7}{2}}W_0^{m+1}E^i_{(22)l} + (h_{00})^{2m+3}W_0E^i_{(31)l}\\
&+ (h_{00})^{m+3}W_0^{2m+2}E^i_{(23)l} + (h_{00})^{m+3}W_0^{2m+1}E^i_{(32)l} + (h_{00})^{m+3}W_0^{2m}E^i_{(41)l}\\
&+ (h_{00}^{\frac{m+5}{2}})W_0^{3m+2}E^i_{(33)l} + (h_{00})^{m+2}W_0^{2m+2}E^i_{(42)l} + (h_{00})^{m+2}W_0^{2m+1}E^i_{(51)l}\\
&+ (h_{00})^2W_0^{4m+2}E^i_{(43)l} + (h_{00})^2W_0^{4m+2}E^i_{(43)l} + (h_{00})^{m+1}W_0^{2m+3}E^i_{(52)l}\\
&+ (h_{00})^2W_0^{4m}E^i_{(61)l} + (h_{00})W_0^{4m+2}E^i_{(62)l} + (h_{00})W_0^{4m+1}E^i_{(7)l} + W_0^{4m+2}E^i_{(8)l}, \quad (4.7)
\end{aligned}$$

where

$$\begin{aligned}
&E^i_{(0)l} = C^i_{(0)r}C^r_{(0)l}, E^i_{(11)l} = C^i_{(11)r}C^r_{(0)l} + C^i_{(0)r}C^r_{(11)l}, E^i_{(21)l} = C^i_{(11)r}C^r_{(11)l},\\
&E^i_{(12)l} = C^i_{(0)r}C^r_{(12)l} + C^i_{(12)r}C^r_{(0)l},\\
&E^i_{(22)l} = C^i_{(12)r}C^r_{(11)l} + C^i_{(11)r}C^r_{(12)l} + C^i_{(21)r}C^r_{(0)l} + C^i_{(0)r}C^r_{(21)l},\\
&E^i_{(31)l} = C^i_{(21)r}C^r_{(11)l} + C^i_{(11)r}C^r_{(21)l}, E^i_{(23)l} = C^i_{(12)r}C^r_{(12)l} + C^i_{(22)r}C^r_{(0)l} + C^i_{(0)r}C^r_{(22)l},\\
&E^i_{(32)l} = C^i_{(21)r}C^r_{(12)l} + C^i_{(12)r}C^r_{(21)l} + C^i_{(3)r}C^r_{(0)l} + C^i_{(22)r}C^r_{(11)l} + C^i_{(11)r}C^r_{(22)l} + C^i_{(0)r}C^r_{(3)l},\\
&E^i_{(41)l} = C^i_{(3)r}C^r_{(11)l} + C^i_{(11)r}C^r_{(3)l} + C^i_{(21)r}C^r_{(21)l}, E^i_{(33)l} = C^i_{(22)r}C^r_{(12)l} + C^i_{(12)r}C^r_{(22)l},\\
&E^i_{(42)l} = C^i_{(4)r}C^r_{(0)l} + C^i_{(3)r}C^r_{(12)l} + C^i_{(22)r}C^r_{(21)l} + C^i_{(21)r}C^r_{(22)l} + C^i_{(12)r}C^r_{(3)l} + C^i_{(0)r}C^r_{(4)l},\\
&E^i_{(51)l} = C^i_{(3)r}C^r_{(21)l} + C^i_{(21)r}C^r_{(3)l} + C^i_{(4)r}C^r_{(11)l} + C^i_{(11)r}C^r_{(4)l}, E^i_{(43)l} = C^i_{(22)r}C^r_{(22)l},\\
&E^i_{(52)l} = C^i_{(4)r}C^r_{(12)l} + C^i_{(12)r}C^r_{(4)l} + C^i_{(3)r}C^r_{(22)l} + C^i_{(22)r}C^r_{(3)l},\\
&E^i_{(61)l} = C^i_{(3)r}C^r_{(3)l} + C^i_{(4)r}C^r_{(21)l} + C^i_{(21)r}C^r_{(4)l}, E^i_{(62)l} = C^i_{(4)r}C^r_{(22)l} + C^i_{(22)r}C^r_{(4)l},\\
&E^i_{(7)l} = C^i_{(4)r}C^r_{(3)l} + C^i_{(3)r}C^r_{(4)l}, E^i_{(8)l} = C^i_{(4)r}C^r_{(4)l}.
\end{aligned}$$

(5). The Calculation for $\Phi^r {\Phi_r}^i{}_l$.
Differentiating (4.5) by y^r, we get

$$4h_{00}^{m+3}W_0^{2m+2}\Phi^r {\Phi_r}^i{}_l = h_{00}^{2m+4}W_0 J^i_{(11)l} + h_{00}^{2m+4}J^i_{(21)l} + h_{00}^{2m+3}W_0^{m+2}J^i_{(12)l} + h_{00}^{2m+3}W_0^{m+1}J^i_{(22)l} + h_{00}^{2m+3}W_0 J^i_{(31)l} + h_{00}^{2m+3}W_0^{2m+2}J^i_{(23)l} + h_{00}^{\frac{3m+5}{2}}W_0^{2m+1}J^i_{(32)l} + h_{00}^{m+3}W_0^{m+1}J^i_{(41)l} + h_{00}^{\frac{3m+5}{2}}W_0^{3m+2}J^i_{(33)l} + h_{00}^{m+2}W_0^{2m+2}J^i_{(42)l} + h_{00}^{m+2}W_0^{m+2}J^i_{(51)l} + h_{00}^{m+1}W_0^{4m+2}J^i_{(43)l} + h_{00}^{\frac{m+3}{2}}W_0^{3m+2}J^i_{(52)l} + h_{00}^{\frac{m+3}{2}}W_0^{2m+2}J^i_{(61)l} + h_{00}W_0^{3m+2}J^i_{(71)l} + W_0^{4m+2}J^i_{(8)l}. \tag{4.8}$$

(6). The main relation
Multiplying equation (4.3) by $h_{00}^{m+2}W_0^{2m+2}$, we have the equality

$$2^{2m}Kh_{00}^{2m+4}W_0^{m+1}h^i_l = 2^{2m+2}h_{00}^{2m+2}W_0^{2m+2} + {}^hR_0{}^i{}_{0l}2^{2m+1}h_{00}^{2m+1}W_0^{m+1}.2^{2m}W_0^{m+1}\Phi^i_{\|l} - 2^{2m}h_{00}^{m+1}W_0.2^{2m}h_{00}^{2m}W_0^{2m+1}\Phi^i_{l\|0} + 2^{2m+3}h_{00}^{2m+2}W_0^{2m+2}\Phi^r {\Phi_r}^i{}_l,$$

where $h^i{}_l = \delta^i{}_l - l^i l_l$. Putting the values of $\Phi^i_{\|l}, \Phi^i_l, \Phi^i_{l\|0}, \Phi^r {\Phi_r}^i{}_l, \Phi^r{}_l \Phi^i{}_r$ in the above equality, by straight forward computation, we finally obtain

$$h_{00}^{2m+2}{\gamma_1}^i{}_{(2m+3)l} + h_{00}^{m+1}{\gamma_2}^i{}_{(2m+7)l} + W_0^{2m+2}{\gamma_3}^i{}_{(2m+7)l} = 0, \tag{4.9}$$

where ${\gamma_1}^i{}_{(2m+3)l}, {\gamma_2}^i{}_{(2m+7)l}$ and ${\gamma_3}^i{}_{(2m+7)l}$ are homogeneous polynomials of degree 2m+3, 2m+7 and 2m+7 in y^i respectively. Here, ${\gamma_1}^i{}_{(2m+3)l} = 0$ is called the curvature part, ${\gamma_2}^i{}_{(2m+7)l} = 0$ is called the vanishing part and ${\gamma_3}^i{}_{(2m+7)l} = 0$ the killing part, respectively.

Proposition 4.2. *The necessary and sufficient condition for a Kropina space (M, F) with* $F = \frac{\alpha^{m+1}}{\beta^m} = \frac{(e^{k(x)})^{\frac{m-1}{2}} h_{00}^{\frac{m+1}{2}}}{2^m {W_0}^m}$ to be of constant curvature K is that (4.9) holds good.

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